TO FIND THE AREA COMMON TO TWO INTERSECTING CIRCLES.

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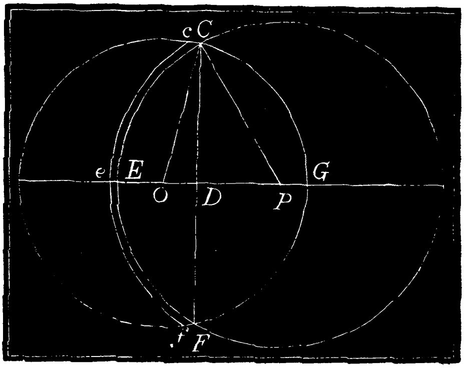

Let O and P be the centers of two intersecting circles. Put $OP = a$, $OC = r$, and suppose the radius CP variable and $= x$. The circles will intersect if x is not greater than $r + a$ nor less than $r - a$. With center P and radius $x + dx$ describe the arc cef indefinitely near CEF. Then

$$DP = \frac{x^2 + a^2 - r^2}{2a}, \text{ arc } CEF = 2x \cos^{-1}\left(\frac{x^2 + a^2 - r^2}{2ax}\right)$$

and the differential of the area $CEFG$ is

$$2x \cos^{-1}\left(\frac{x^2 + a^2 - r^2}{2ax}\right) dx.$$

Putting $\triangle$ for the area sought, we have

$$\triangle = \int_{r-a}^{x} 2x \cos^{-1}\left(\frac{x^2 + a^2 - r^2}{2ax}\right) dx.$$

$$\int 2x \cos^{-1}\left(\frac{x^2 + a^2 - r^2}{2ax}\right) dx = x^2 \cos^{-1}\left(\frac{x^2 + a^2 - r^2}{2ax}\right)$$

$$+ \int \frac{x(r^2 - a^2 + x^2) dx}{\sqrt{4a^2r^2 - (r^2 + a^2 - x^2)^2}}.$$

Put $r^2 + a^2 - x^2 = y$, then

$$\int \frac{x(r^2 - a^2 + x^2) dx}{\sqrt{4a^2r^2 - (r^2 + a^2 - x^2)^2}} = \int \frac{-\frac{1}{2}(2r^2 - y) dy}{\sqrt{4a^2r^2 - y^2}}$$

$$= \int \frac{-r^2 dy}{\sqrt{4a^2r^2 - y^2}} + \int \frac{\frac{1}{2} y dy}{\sqrt{4a^2r^2 - y^2}} = r^2 \cos^{-1}\left(\frac{y}{4ar}\right) - \tfrac{1}{2}\sqrt{4a^2r^2 - y^2}.$$

$$\therefore \triangle = x^2 \cos^{-1}\left(\frac{x^2+a^2-r^2}{2\,a\,x}\right)+r^2\cos^{-1}\left(\frac{a^2+r^2-x^2}{2\,a\,r}\right)-a\sqrt{r^2-\left(\frac{a^2+r^2-x^2}{2\,a}\right)^2}.$$

When $x = R$,

$$\triangle = R^2\cos^{-1}\left(\frac{a^2-r^2+R^2}{2\,a\,R}\right)+r^2\cos^{-1}\left(\frac{a^2+r^2-R^2}{2\,a\,r}\right)$$
$$-a\sqrt{r^2-\left(\frac{a^2+r^2-R^2}{2\,a}\right)^2},$$

which agrees with the result obtained by the ordinary method.

The above formula may be readily adapted to any special case.

When the center P is on the circumference of the other circle, $a = r$, and

$$\triangle = R^2\cos^{-1}\left(\frac{R}{2\,r}\right)+2\,r^2\sin^{-1}\left(\frac{R}{2\,r}\right)-\tfrac{1}{2}R\sqrt{4\,r^2-R^2}.$$

If the circles are equal, $R = r$ and

$$\triangle = 2\,r^2\cos^{-1}\left(\frac{a}{2r}\right)-\tfrac{1}{2}a\sqrt{4\,r^2-a^2}.$$

When they are equal and the center of one on the circumference of the other, $R = r$, $a = r$ and

$$\triangle = r^2\left(\frac{2}{3}\pi-\frac{1}{2}\sqrt{3}\right).$$

PROBLEMS.

5. In a plane triangle there are given the three lines bisecting the angles, a, b and c, to find the sides.—Communicated by Dr. David S. Hart, Stonington. Conn.

6. Find a convenient formula for calculating the capacity of a cistern constructed as follows, viz: Having a lower concavity which is a spherical segment whose versed sine is a and chord $2\,r$, a central cylindrical part whose radius is r and perpendicular height h, and an upper concavity which is a spherical segment whose versed sine is b and chord $2\,r$.—Communicated by Frank Pelton, C. E., Des Moines, Iowa.

7. Multiply $\sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\&c.}}}$ by $\sqrt{1+1\sqrt{1+1\sqrt{1+\&c.}}}$

and express the product in a finite number of terms.—Communicated by Prof. Daniel Kirkwood, Bloomington, Ind.